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Opinion dynamics of stubborn agents under the presence of a troll as differential game

Aykut YILDIZ[1,*], **Arif Bülent ÖZGÜLER**[2]

[1]Department of Electrical and Electronics Engineering, Faculty of Engineering, TED University, Ankara, Turkey
[2]Department of Electrical and Electronics Engineering, Faculty of Engineering, Bilkent University, Ankara, Turkey

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Abstract: The question of whether opinions of stubborn agents result in Nash equilibrium under the presence of troll is investigated in this study. The opinion dynamics is modelled as a differential game played by n agents during a finite time horizon. Two types of agents, ordinary agents and troll, are considered in this game. Troll is treated as a malicious stubborn content maker who disagrees with every other agent. On the other hand, ordinary agents maintain cooperative communication with other ordinary agents and they disagree with the troll. Under this scenario, explicit expressions of opinion trajectories are obtained by applying Pontryagin's principle on the cost function. This approach provides insight into the social networks that comprise a troll in addition to ordinary agents.

Key words: Opinion dynamics, social network, differential game, Nash equilibrium, Pontryagin's principle, troll

1. Introduction

Opinion dynamics is defined as the study of how large groups interact with each other and reach consensus [1]. Although research on opinion dynamics dates back to 50s such as [2], the topic has been booming in the past decade owing to the rise of the social networks. The agent based models of social networks discussed in the survey [3] is one of the hottest topics that the control theory community is focusing on. In addition to social networks, opinion dynamics has numerous applications such as jury panels, government cabinets, and company board of directors as noted in [3].

Naive approach on modelling opinion dynamics is [4] where exact consensus is shown to occur if the graph of network is strongly connected. This notion is transcended to partial consensus under the presence of stubborn agents in [5]. The study on stubbornness is extended to relatively more sophisticated network topologies such as Erdos–Renyi random graphs and small-world graphs in [6]. A nonlinear attraction force is considered on top of linear stubbornness force in [7].

The disagreements in social networks have been studied extensively in the opinion dynamics literature. The origin of disagreement in the network is declared as culture, ethnicity, or religion in [8]. On the other hand, origin of disagreement is assumed to be competition among the agents in [9] and [10]. The question of whether cooperation can result from such a competition is answered in these studies as well. For a comprehensive survey on origins of cooperation and competition among human beings, you may see [11]. The disagreements among the agents have been modelled as antagonistic interactions in [12] and [13]. In so-called Altafini model,

*Correspondence: aykut.yildiz@tedu.edu.tr

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negative edge weights are utilized for antagonistic interactions, and consensus occurs on two separate positive and negative opinions [14–17]. It is shown that disagreements result in clusters of opinions in [18–20]. Similar to our study, the disagreements are modelled as repulsion between the agents in [21] and disagreements are shown to result in oscillations of opinions in [22]. However, explicit trajectories are not evaluated in these methods, which distinguishes it from our method.

Our main contribution is to establish that opinion transactions in a social network can be modeled as a differential game under the presence of a troll. Here, troll is regarded as a malicious content maker in the social network and he is a stubborn agent who disagrees with everyone and with whom everybody disagrees. Another study which focuses on differential game of opinions in social networks is [23]. Here, this notion is extended to the social networks which comprise a troll in addition to ordinary agents. Explicit expressions of opinions are derived for such a scenario by using Pontryagin's principle based on [24]. Such a game theoretical model of social networks is useful since it provides a rigorous mathematical tool which provides a deeper understanding of opinion dynamics under the presence of a troll.

The paper is organized as follows. The differential game based optimization problem of opinion dynamics is introduced in Section 2. The main theorem on Nash equilibrium and the resulting opinion trajectories is presented in Section 3. An example of dispute on a topic in social networks is argued in Section 4. Conclusions and future works are discussed in Section 5. Finally, the appendix is dedicated to the comprehensive derivation of explicit expressions of opinion trajectories.

2. Problem definition

Our objective is to model opinion dynamics of a social network as a differential game played by a troll in addition to $n-1$ ordinary agents. This problem is crucial since it provides insight into the dynamics of opinions by using rigorous differential games and Nash equilibrium concepts. The cost functionals of the troll and ordinary agents in this game are respectively,

$$J_1(\mathbf{x}, b_1, u_1) = \frac{1}{2}\int_0^\tau \{w_{11}(x_1 - b_1)^2 + u_1^2 - \sum_{j\in\{N-\{1\}\}} p_j(x_1 - x_j)^2\}dt, \tag{1}$$

and

$$J_i(\mathbf{x}, b_i, u_i) = \frac{1}{2}\int_0^\tau \{w_{ii}(x_i - b_i)^2 + u_i^2 - r_i(x_i - x_1)^2 + \sum_{j\in\{N-\{1,i\}\}} w_{ij}(x_i - x_j)^2\}dt \quad for\ i = 2, 3, ..., n, \tag{2}$$

where agent 1 is the troll and the other $n-1$ agents are ordinary. J_i is the cost functional minimized by the i^{th} agent. The quantities $b_i = x_i(0)$, and $x_i(t)$ are the initial and instantaneous opinions of agents, respectively. The vector with $x_i(t)$ at the i^{th} entry is denoted by $\mathbf{x}(t)$, which thus represents all opinions at time t. During the game, the i^{th} agent commands $u_i(t)$, its control input at t. The duration of the game of information transaction is fixed and it is equal to τ. The constant w_{ii} is the stubbornness coefficient of i^{th} agent and w_{ij} represents the influence of j^{th} agent on the i^{th} agent. The constant p_j measures the repulsion of j^{th} agent to the troll when positive and r_i, the repulsion of troll to the i^{th} agent. Also, let N denote the set of agents $N = \{1, 2, ..., n\}$ which is fixed throughout the game. It will be assumed that all real numbers w_{ij}, r_i, p_j are nonnegative so that there is repulsion between troll and ordinary agents. r_i, p_j will occasionally be allowed to

be negative as well, in order to be able to compare this game with a previously considered game in [23]. The technical analysis below will be valid for $r_i,\ p_j \in \mathbb{R}$ although our focus is on the case $r_i, p_j \geq 0$ as our main objective is to investigate networks with a troll.

The first components in the integrals of (1) and (2) represent the stubbornness of agents and, the second, their cumulative control efforts. The third components measure the cumulative disagreement between the troll and the ordinary agents, and the last components in (2) stand for the influence among the ordinary agents. To sum up, the troll is modelled as a stubborn agent who disagrees with other agents and with whom the other agents disagree, but allow mutual positive as well as negative influences. Under this scenario, the game played by the agents is

$$\min_{u_i}\{J_i\} \ subject\ to\ \dot{x_i} = u_i \qquad for\ i = 1, 2, ..., n, \tag{3}$$

so that the agents control their rate of change of opinion and thereby try to minimize their individual costs of holding an opinion.

This game is similar to that in [23] with the significant difference of existence of a troll. This brings in a brand new technical dimension to the game as it makes the cost functionals nonconvex. The troll disagrees with ordinary agents via the r_i coefficients, and the ordinary agents disagree with the troll via p_j coefficients. This provides a new degree of freedom in the social network as, in the default case when r_i, p_j's are nonnegative, varying degrees of repulsion between the troll and the ordinary agents can be examined for its effect on the evolution of opinions. It is assumed that there is a single troll and single opinion, but these can be generalized to higher dimensions trivially.

Obtaining the opinion trajectories of the differential game in (3) is a comprehensive task which requires the following step by step approach. First of all, the cost functions in (1) and (2) are converted to Hamiltonians with ease. Secondly, the Pontryagin's principle is used for evaluating the ordinary differential equations for those Hamiltonians. Those differential equations are transformed to state equations by a straightforward substitution of variables. The problem that we obtain is an LTI boundary value problem whose closed form solution is of interest. In order to convert the boundary value problem to initial value problem, the unspecified terminal condition in Pontryagin's principle is imposed. The solution to the resulting initial value problem is determined in terms of blocks of state transition matrix. By substituting the matrix functions into those blocks, the eventual explicit expressions of opinion trajectories are calculated.

3. Main results

In this section, the main theorem on the opinion trajectories is presented. The extensive derivation of opinion trajectories is left to the appendix.

Suppose that the entries of $\mathbf{s}$ vector are given by

$$s_i = w_{ii} b_i \qquad\qquad for\ \ i = 1, 2, ..., n,$$

and let

$$Q = \begin{bmatrix} q_{11} & p_2 & p_3 & \cdots & p_n \\ r_2 & q_{22} & -w_{23} & \cdots & -w_{2n} \\ r_3 & -w_{32} & q_{33} & & \\ \vdots & \vdots & & \ddots & \\ r_n & -w_{n2} & & \cdots & q_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{4}$$

where

$$\begin{array}{ll} q_{11} & = w_{11} - p_2 - p_3 - \cdots - p_n \\ q_{22} & = -r_2 + w_{22} + w_{23} + \cdots + w_{2n} \\ q_{33} & = -r_3 + w_{32} + w_{33} + \cdots + w_{3n} \\ \vdots & = \qquad \vdots \\ q_{nn} & = -r_n + w_{n2} + w_{n3} + \cdots + w_{nn}. \end{array} \tag{5}$$

Theorem 1 *Consider the game (1)-(3). Let Q be nonsingular.*

(i) A necessary condition for a Nash equilibrium to exist in the interval $[0, \tau)$ is that Q does not have a negative eigenvalue $-r^2$ satisfying $r = (2k+1)\frac{\pi}{2\tau}$ for any integer k.

(ii) If (i) holds, then the opinion trajectories of any Nash equilibrium are given by $x_j(t); t \in [0, \tau)$, $j = 1, ..., n$, where with $\mathbf{x} = [x_1, ..., x_n]^T$

$$\begin{array}{rl} \mathbf{x}(t) = & \{cosh(\sqrt{Q}t) - sinh(\sqrt{Q}t)cosh(\sqrt{Q}\tau)^{-1}sinh(\sqrt{Q}\tau)\}\mathbf{b} \\ & +\{(I - cosh(\sqrt{Q}t))Q^{-1} + sinh(\sqrt{Q}t)Q^{-1}cosh(\sqrt{Q}\tau)^{-1}sinh(\sqrt{Q}\tau)\}\mathbf{s}, \end{array} \tag{6}$$

for a square root $\sqrt{Q}$ of Q.

Remark 1 *The condition ii states that if the Nash equilibrium of the game (1)-(3) exists, then it is necessarily in the form subscribed by $\mathbf{x}(t)$ in (6). The opinion trajectory (6) expresses the evolution of the opinions of n-agents starting from the initial opinions b_i's. The opinion $x_i(t)$ at time t of agent-i is dependent on the initial opinions of all agents. This necessitates that the Nash equilibrium opinion trajectories are expressed in a vector form, i.e. in a coupled or interactive expression (6). In certain special cases it is possible to express the Nash opinion trajectories of each agent in a decoupled form [23].*

Remark 2 *Since the individual cost functions (1), (2) are not in general convex, the fact that the given solution is indeed a Nash equilibrium is not easy to establish. However, the special cases examined in Corollary 1 strongly indicate that this is plausible.*

Remark 3 *A more compact expression for (6) is obtained with $W := [w_{ij}]$ as*

$$\mathbf{x}(t) = \{Q^{-1}W + cosh[H(\tau - t)]cosh(H\tau)^{-1}(I - Q^{-1}W)\}\mathbf{b} \tag{7}$$

where H is the square root of Q. This expression at $t = \tau$ can be used to obtain the disparity, or distance, among opinions at the end of the interval of interaction.

Corollary 1 *If in (1) and (2), $p_j = -w_{1j}$, $r_j = -w_{j1}$ for $j = 2, ..., n$ for positive w_{1j}, w_{j1}, then a Nash equilibrium exists and is unique.*

Remark 4 *Note that the existence and uniqueness of Nash equilibrium occurs in this special case, where the troll conforms to the society. Such a Nash equilibrium has been examined in detail in [23] with its multivariable (multiopinion) extension given in [25].*

Remark 5 *If Q has a negative eigenvalue $-r^2$ such that r is not an odd multiple of $\frac{\pi}{2\tau}$, then some entries of $\mathbf{x}(t)$ are oscillatory. As τ gets closer to a value so as to have $r = (2k+1)\frac{\pi}{2\tau}$ for some integer k, then the amplitude of oscillation gets larger to eventually prohibit the existence of an equilibrium.*

4. Application example

In this section, three examples are presented where the issue is the punishment for violence to women. A large positive opinion indicates that the violence to women should be punished severely whereas a large negative opinion indicates that violence to women is favorable. In order to understand the mechanism of such a discussion, three experiments are constructed as follows. The parameters of those experiments are listed in Tables 1 and 2. In these tables, $\mathcal{U}(a,b)$ stands for uniformly distributed random variable between a and b.

Table 1. Parameter values for Figure 1.

Figure 1-Case 1	Figure 1-Case 2	Figure 1-Case 3
$n = 50$ agents	$n = 50$ agents	$n = 50$ agents
$\tau = 2\ s$	$\tau = 2\ s$	$\tau = 2\ s$
$T_s = 0.001\ s$	$T_s = 0.001\ s$	$T_s = 0.001\ s$
$p_j = 0 \quad for \quad j = 2,3,...,n$	$p_j = 0 \quad for \quad j = 2,3,...,n$	$p_j = 0 \quad for \quad j = 2,3,...,n$
$r_i = 0 \quad for \quad i = 2,3,...,n$	$r_i = 0 \quad for \quad i = 2,3,...,n$	$r_i = 0 \quad for \quad i = 2,3,...,n$
$w_{11} = 6$	$w_{11} = 6$	$w_{11} = 6$
$w_{ij} \sim \mathcal{U}(0,0.1) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$	$w_{ij} \sim \mathcal{U}(0,0.2) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$	$w_{ij} \sim \mathcal{U}(0,0.3) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$
$b_1 = -10$	$b_1 = -10$	$b_1 = -10$
$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$	$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$	$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$

Table 2. Parameter values for Figure 2.

Figure 2-Case 1	Figure 2-Case 2	Figure 2-Case 3
$n = 50$ agents	$n = 50$ agents	$n = 50$ agents
$\tau = 2\ s$	$\tau = 2\ s$	$\tau = 2\ s$
$T_s = 0.001\ s$	$T_s = 0.001\ s$	$T_s = 0.001\ s$
$p_j \sim \mathcal{U}(0,5) \quad for \quad j = 2,3,...,n$	$p_j \sim \mathcal{U}(0,5) \quad for \quad j = 2,3,...,n$	$p_j \sim \mathcal{U}(0,5) \quad for \quad j = 2,3,...,n$
$r_i \sim \mathcal{U}(0,5) \quad for \quad i = 2,3,...,n$	$r_i \sim \mathcal{U}(0,5) \quad for \quad i = 2,3,...,n$	$r_i \sim \mathcal{U}(0,5) \quad for \quad i = 2,3,...,n$
$w_{11} = 6$	$w_{11} = 6$	$w_{11} = 6$
$w_{ij} \sim \mathcal{U}(0,0.2) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$	$w_{ij} \sim \mathcal{U}(0,0.4) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$	$w_{ij} \sim \mathcal{U}(0,0.6) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$
$b_1 = -10$	$b_1 = -10$	$b_1 = -10$
$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$	$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$	$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$
Figure 2-Case 4	Figure 2-Case 5	Figure 2-Case 6
$n = 50$ agents	$n = 50$ agents	$n = 50$ agents
$\tau = 2\ s$	$\tau = 2\ s$	$\tau = 2\ s$
$T_s = 0.001\ s$	$T_s = 0.001\ s$	$T_s = 0.001\ s$
$p_j \sim \mathcal{U}(5,10) \quad for \quad j = 2,3,...,n$	$p_j \sim \mathcal{U}(5,10) \quad for \quad j = 2,3,...,n$	$p_j \sim \mathcal{U}(5,10) \quad for \quad j = 2,3,...,n$
$r_i \sim \mathcal{U}(5,10) \quad for \quad i = 2,3,...,n$	$r_i \sim \mathcal{U}(5,10) \quad for \quad i = 2,3,...,n$	$r_i \sim \mathcal{U}(5,10) \quad for \quad i = 2,3,...,n$
$w_{11} = 6$	$w_{11} = 6$	$w_{11} = 6$
$w_{ij} \sim \mathcal{U}(0,0.2) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$	$w_{ij} \sim \mathcal{U}(0,0.4) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$	$w_{ij} \sim \mathcal{U}(0,0.6) \quad for \quad i = 2,3,...,n$ $for \quad j = 2,3,...,n$
$b_1 = -10$	$b_1 = -10$	$b_1 = -10$
$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$	$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$	$b_i \sim \mathcal{U}(0,40) \quad for \quad i = 2,3,...,n$

In Figure 1, the case where there is no interaction between troll and ordinary agents is investigated. In other words, the only communication between the troll and ordinary agents, i.e. repulsion is considered as zero in the first experiment. In this case, the opinion of troll does not change since he is stubborn and attains constant opinion. On the other hand, there is intensive interaction among the ordinary agents which drives the system towards consensus. As the number of ordinary agents or w_{ij} parameters in (2) increase, exact consensus occurs at the average of initial opinions,i.e. $\mathbf{x}(\tau) = \mathbf{20}$. The case where there is attraction between the troll and ordinary agents can also be considered by setting p_j and r_i in (1) and (2) to negative values. Then, the first agent will be partial troll who sometimes claims plausible arguments and conforms to society.

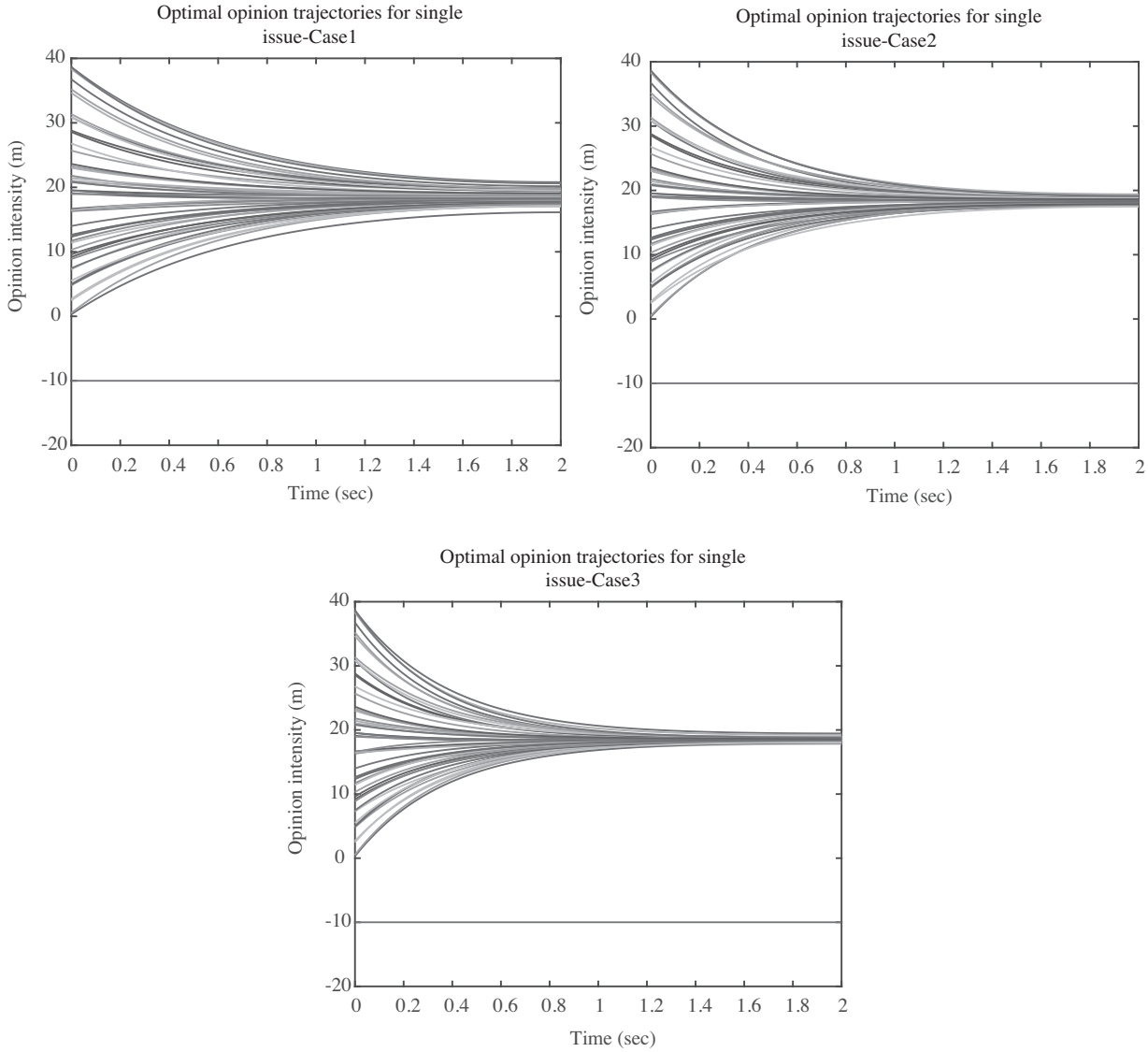


Figure 1. Optimal opinion trajectories for no repulsion between troll and ordinary agents during the game of opinion transactions: this illustration shows that the troll will not change his opinion if the repulsion parameter is zero, as the mere interaction between troll and ordinary agents is via the repulsion parameter.

In Figure 2, it is observed that the initial opinion of troll is negative where he claims that women deserve violence. Then, a reaction arises from the network which results in alternations of the opinion of troll where he

attains negative and positive opinions periodically. Such alternations are a typical feature of trolls since they are more inconsistent compared to the ordinary agents. The alternations emerge because the troll regrets his initial strange opinion and temporarily conforms to society. He apologizes and adopts a reasonable opinion; however, the strange opinions emerge after some time. The frequency of alternations which represent the intensity of inconsistency increases as repulsion parameters increase. The opinion trajectories of ordinary agents reveal that they are more consistent compared to the troll. Their opinions exhibit a consensus towards a positive value of the issue that is considered here, namely violence to women. Therefore, they consistently claim that the violence to women should be punished throughout the excessive transactions of opinions.

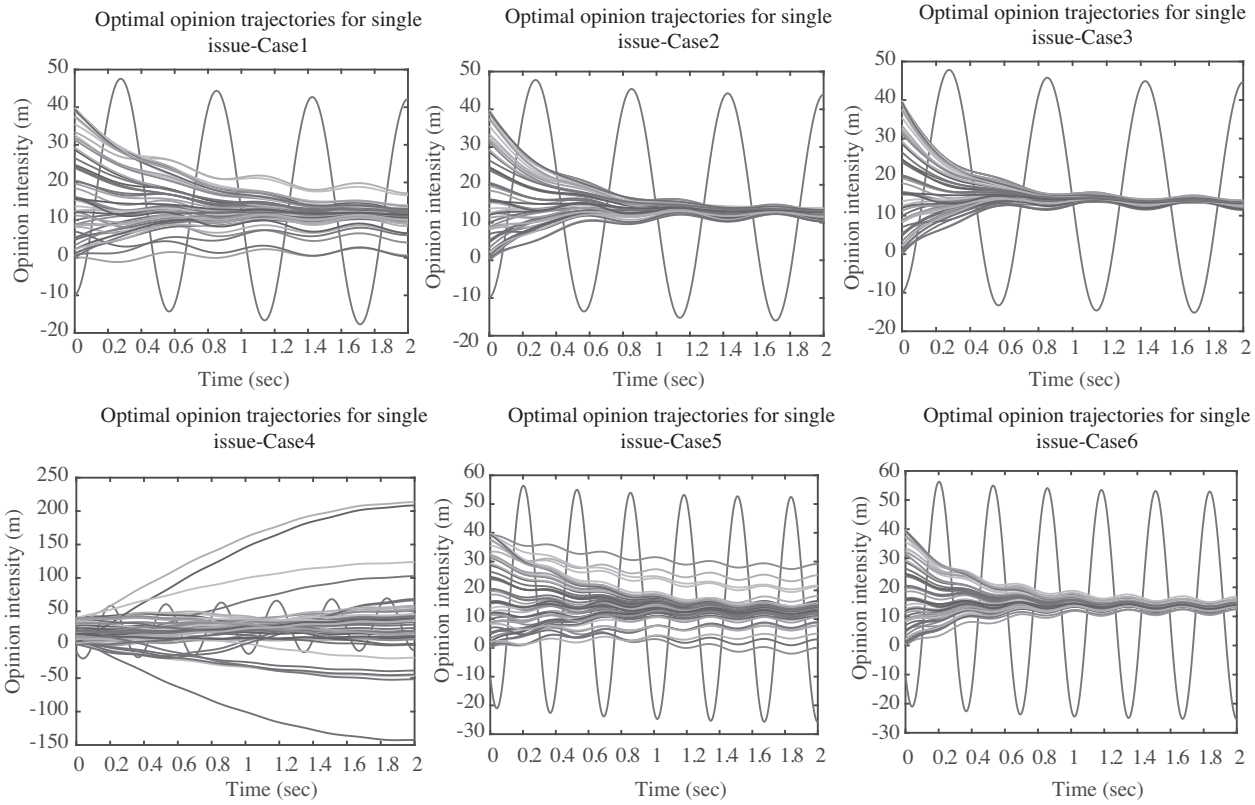


Figure 2. We visualize the optimal opinion trajectories for various parameters here. This illustrates the fluctuations of opinion of troll due to his underlying inconsistency. No matter how the troll behaves, the ordinary agents exhibit a cooperative communication which results in consensus except in Case 4. This case stands out because the repulsion parameter in this case dominates the influence parameters that have smaller values than in other cases.

In Figure 3, our main objective is to illustrate the case where item (i) in Theorem 1 is violated. In other words, the opinion exchange duration τ is allowed to get close to $\frac{\pi}{2r}$ where $-r^2$ is a negative eigenvalue of Q matrix in (4). In this experiment, the number of agents is selected as $n = 20$ and the sampling period is equal to $T_s = 0.001$. The p_j parameters in (1) and r_i parameters in (2) are selected as uniformly distributed between $[0, 5]$. The w_{11} parameter is chosen as 6 and the w entries in (5) are selected as uniformly distributed between $[0, 0.03]$. The initial opinions b_i in (2) are assigned as uniformly distributed between $[0, 15]$. For these parameter selections, Q matrix in (4) has a negative eigenvalue $\lambda_1 = -56.523$. The r parameter in item (i) of Theorem 1 is equal to $r = \sqrt{-\lambda_1}$ which corresponds to $r = 7.518$. Thus, the game duration $\tau = \frac{\pi}{2r}$ turns out

to be $\tau = 0.209$. Under these selections of parameters, it is expected that some of the opinion intensities will blow up to large unstable values according to item (i) of Theorem 1. In Figure 3, it is indeed observed that the opinion intensity of troll assume large values under this scenario.

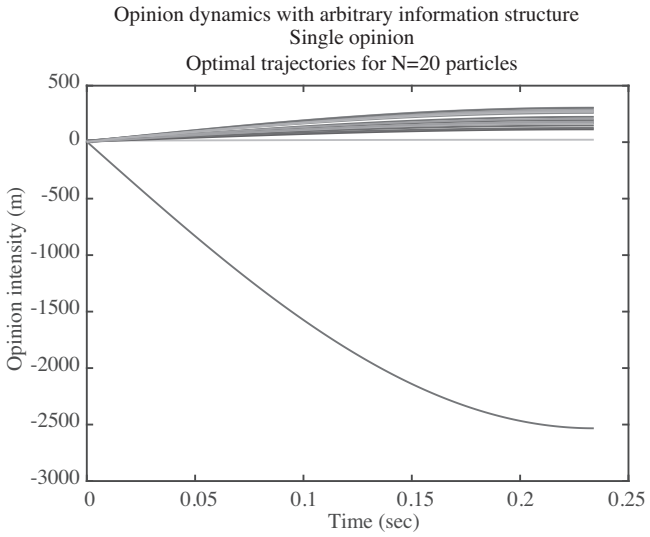


Figure 3. Approximately unstable case is shown for optimal opinion trajectories in which game duration τ is allowed to get close to $\frac{\pi}{2r}$ where $-r^2$ is a negative eigenvalue of Q in (4). This displays the case where the opinions of troll blow up to infinity while concentrating on disagreeing with the ordinary agents. The ordinary agents are not adversely affected by this polarization due to their substantial momentum.

5. Conclusions

In this study, the extension of [23] to networks with a troll is discussed. This corresponds to the case where certain interaction coefficients in (1) and (2) are repulsive and thus have a minus sign. If those coefficients are positive, then this boils down to [23] where the solution represents a Nash equilibrium. The fact that the cost functions are nonconvex presents a challenge to establish the sufficiency of the condition (i) of Theorem 1. Nevertheless, the Nash equilibrium, if it exists, is included in the set of opinion dynamics described by condition (ii) of Theorem 1.

An extension to multiple issues is in a manner similar to the extension of [23] to [25]. We have considered in (1) and (2), the unspecified terminal condition case. Alternatives such as specified or free terminal conditions also need to be examined and may model different ideologies in societies. Finally, the perfect integrator controls of agents in (3), replaced with more general, still linear, control models may also be explored.

Appendix

Here, the necessary conditions in Section 6.5.1 of [24] are employed in order to determine the explicit expression (6) in Theorem 1. Since there are two types of agents, namely troll and ordinary agents, we thus have two different Hamiltonians given by

$$H_1 = \frac{1}{2}\{w_{11}(x_1 - b_1)^2 + u_1^2 - \sum_{j\in\{N-\{1\}\}} p_j(x_1 - x_j)^2\} + \rho_1 u_1,$$

and

$$H_i = \frac{1}{2}\{w_{ii}(x_i - b_i)^2 + u_i^2 - r_i(x_i - x_1)^2 + \sum_{j\in\{N-\{1,i\}\}} w_{ij}(x_i - x_j)^2\} + \rho_i u_i \quad for \quad i = 2,3,...,n,$$

where ρ_i is the costate of i^{th} agent. The other parameters of these expressions are defined in Section 2 after (1) and (2). A set of ordinary differential equations are obtained by applying the rules $\frac{\partial H^i}{\partial u^i} = 0$, $\dot{\rho}_i = -\frac{\partial H^i}{\partial x^i}$, on the Hamiltonians as

$$\begin{aligned} u_i &= -\rho_i, && for \quad i = 1,2,...,n \\ \dot{\rho}_1 &= -\{w_{11}(x_1 - b_1) - \sum_{j\in\{N-\{1\}\}} p_j(x_1 - x_j)\}, \\ \dot{\rho}_i &= -\{w_{ii}(x_i - b_i) - r_i(x_i - x_1) + \sum_{j\in\{N-\{1,i\}\}} w_{ij}(x_i - x_j)\} && for \quad i = 2,3,...,n \\ \dot{x}_i &= u_i && for \quad i = 1,2,...,n. \end{aligned} \tag{8}$$

$$\rho_i(\tau) = 0 \qquad for \quad i = 1,2,...,n. \tag{9}$$

The last boundary condition is known as the unspecified terminal condition in optimal control terminology. The differential equations in (8) can be written in compact form as the following state equation

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ -Q & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \rho(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{s} \end{bmatrix}, \tag{10}$$

where $\mathbf{x} := [\ x_1,...,x_n\ ]'$, $\rho := [\ \rho_1,...,\rho_n\ ]'$, $\mathbf{s} := [\ s_1,...,s_n\ ]'$ and $Q \in \mathbb{R}^{n\times n}$. The entries of $\mathbf{s}$ vector are given by

$$s_i = w_{ii}b_i \qquad for \quad i = 1,2,...,n.$$

where w_{ii} and b_i are introduced after (2).

The Q matrix in (10) can be written explicitly as (4) where the diagonal entries are given by (5).

The solution of the LTI system in (10) is determined as

$$\begin{bmatrix} \mathbf{x}(t) \\ \rho(t) \end{bmatrix} = \phi(t) \begin{bmatrix} \mathbf{b} \\ \rho(0) \end{bmatrix} + \psi(t,0)\mathbf{s}. \tag{11}$$

Here, $\psi(t,0) \in \mathbb{R}^{2n\times n}$ and state transition matrix $\phi(t) \in \mathbb{R}^{2n\times 2n}$ can be computed in Laplace transform domain as

$$\begin{aligned} \phi(t) &= \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} := \mathcal{L}^{-1}\{\begin{bmatrix} sI & I \\ Q & sI \end{bmatrix}^{-1}\}, \\ \psi(t,t_0) &:= \int_{t_0}^{t} \begin{bmatrix} \phi_{12}(t-\tilde{\tau}) \\ \phi_{22}(t-\tilde{\tau}) \end{bmatrix} d\tilde{\tau}, \end{aligned} \tag{12}$$

where state transition matrix blocks $\phi_{ij}(t) \in \mathbb{R}^{n\times n}$. The matrix inversion above is calculated using block matrices as

$$\begin{bmatrix} sI & I \\ Q & sI \end{bmatrix}^{-1} = \begin{bmatrix} s(s^2I - Q)^{-1} & -(s^2I - Q)^{-1} \\ -Q(s^2I - Q)^{-1} & s(s^2I - Q)^{-1} \end{bmatrix}.$$

The blocks of state transition matrix $\phi_{ij}(t)$ can be obtained using inverse Laplace transform which gives

$$\begin{aligned}\phi_{11}(t) &= \phi_{22}(t) = cosh(\sqrt{Q}t)\\ \phi_{12}(t) &= -sinh(\sqrt{Q}t)(\sqrt{Q})^{-1}\\ \phi_{21}(t) &= -\sqrt{Q}sinh(\sqrt{Q}t),\end{aligned} \tag{13}$$

$$\begin{aligned}\psi_1(t,0) &= (I - cosh(\sqrt{Q}t))Q^{-1}\\ \psi_2(t,0) &= sinh(\sqrt{Q}t)(\sqrt{Q})^{-1},\end{aligned} \tag{14}$$

where $\psi_i(t,0) \in \mathbb{R}^{n\times n}$. The initial costate $\rho(0)$ can be obtained by imposing the boundary condition in (9) on the solution in (12)

$$\rho(\tau) = \phi_{21}(\tau)\mathbf{b} + \phi_{22}(\tau)\rho(0) + \psi_2(\tau,0)\mathbf{s}.$$

Thus, the boundary value problem in (8) and (9) can be converted to an initial value problem by using the above relation. The initial costate $\rho(0)$ above can be plugged into the solution in (12) to obtain the opinion trajectories as

$$\begin{aligned}\mathbf{x}(t) = \quad & \{\phi_{11}(t) - \phi_{12}(t)\phi_{22}(\tau)^{-1}\phi_{21}(\tau)\}\mathbf{b}\\ & +\{\psi_1(t,0) - \phi_{12}(t)\phi_{22}(\tau)^{-1}\psi_2(\tau,0)\}\mathbf{s},\end{aligned} \tag{15}$$

provided $\phi_{22}(\tau)^{-1}$ exists. This is the case if and only if $\phi_{22}(t) = cosh(\sqrt{Q}t)$ is nonsingular where $\sqrt{Q}$ is a possibly nonreal square root of Q. This in turn is equivalent to condition (i) of Theorem 1, by [25]. The necessity of the condition (i) is thus established.

If the matrix blocks in (13) and (14) are plugged into (15), the explicit solution can be obtained for the opinion trajectories as

$$\begin{aligned}\mathbf{x}(t) = \quad & \{cosh(\sqrt{Q}t) - sinh(\sqrt{Q}t)cosh(\sqrt{Q}\tau)^{-1}sinh(\sqrt{Q}\tau)\}\mathbf{b}\\ & +\{(I - cosh(\sqrt{Q}t))Q^{-1} + sinh(\sqrt{Q}t)Q^{-1}cosh(\sqrt{Q}\tau)^{-1}sinh(\sqrt{Q}\tau)\}\mathbf{s}.\end{aligned}$$

This proves the condition (ii). Note that under the circumstance of Remark 5, $\sqrt{Q}$ will be complex in general. This expression will still result in an opinion trajectory with real entries because $\mathbf{x}(\mathbf{t})$ is a function of Q, i.e. an even function of $\sqrt{Q}$.

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